

Uniform rigidity sequences for weak mixing diffeomorphisms on \mathbb{T}^2

PHILIPP KUNDE

November 12, 2014

Abstract

In this paper we will show that if a sequence of natural numbers satisfies a certain growth rate, then there is a weak mixing diffeomorphism on \mathbb{T}^2 that is uniformly rigid with respect to that sequence. The proof is based on a quantitative version of the Anosov-Katok-method with explicitly defined conjugation maps and the constructions are done in the C^∞ -topology as well as in the real-analytic topology.

1 Introduction

In [GM89] the notion of uniform rigidity was introduced as the topological analogue of rigidity in ergodic theory:

- Definition 1.1.**
1. Let T be an invertible measure-preserving transformation of a non-atomic probability space (X, \mathcal{B}, μ) . T is called rigid if there exists an increasing sequence $(n_m)_{m \in \mathbb{N}}$ of natural numbers such that the powers T^{n_m} converge to the identity in the strong operator topology as $m \rightarrow \infty$, i.e. $\|f \circ T^{n_m} - f\|_2 \rightarrow 0$ as $m \rightarrow \infty$ for all $f \in L^2(X, \mu)$. So rigidity along a sequence $(n_m)_{m \in \mathbb{N}}$ implies $\mu(T^{n_m} A \cap A) \rightarrow \mu(A)$ as $m \rightarrow \infty$ for all $A \in \mathcal{B}$.
 2. Let (X, \mathcal{B}, μ) be a Lebesgue probability space, where X is a compact metric space with metric d . A measure-preserving homeomorphism $T : X \rightarrow X$ is called uniformly rigid if there exists an increasing sequence $(n_m)_{m \in \mathbb{N}}$ of natural numbers such that $d_u(T^{n_m}, id) \rightarrow 0$ as $m \rightarrow \infty$, where $d_u(S, T) = d_0(S, T) + d_0(S^{-1}, T^{-1})$ with $d_0(S, T) := \sup_{x \in X} d(S(x), T(x))$ is the uniform metric on the group of measure-preserving homeomorphisms on X .

Remark 1.2. Uniform rigidity implies rigidity. In [Ya13], example 3.1, an example of a rigid, but not uniformly rigid homeomorphism of \mathbb{T}^2 is presented. Thus, rigidity and uniform rigidity do not coincide on \mathbb{T}^2 .

In [JKLSS09], Proposition 4.1., it is shown that if an ergodic map is uniformly rigid, then any uniform rigidity sequence has zero density. Afterwards, the following question is posed:

Question 1.3. Which zero density sequences occur as uniform rigidity sequences for an ergodic transformation?

Under some assumptions on the sequence $(n_m)_{m \in \mathbb{N}}$ measure-preserving transformations that are weak mixing and rigid along this sequence are constructed by a cutting and stacking method in [BJLR]. Recall that a measure-preserving transformation $T : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$ is called weak mixing if for all $A, B \in \mathcal{B}$: $\frac{1}{N} \sum_{n=1}^N |\mu(T^n A \cap B) - \mu(A) \cdot \mu(B)| \rightarrow 0$ as $N \rightarrow \infty$.

K. Yancey considered Question 1.3 in the setting of homeomorphisms on \mathbb{T}^2 (see [Ya13]). Given a sufficient growth rate of the sequence she proved the existence of a weak mixing homeomorphism of \mathbb{T}^2 that is uniformly rigid with respect to this sequence: Let $\psi(x) = x^3$. If $(n_m)_{m \in \mathbb{N}}$ is an increasing sequence of natural numbers satisfying $\frac{n_{m+1}}{n_m} \geq \psi(n_m)$, there exists a weak mixing homeomorphism of \mathbb{T}^2 that is uniformly rigid with respect to $(n_m)_{m \in \mathbb{N}}$.

In this paper we start to examine this problem in the smooth category. The aimed diffeomorphisms are constructed with the aid of the so-called “conjugation by approximation-method” introduced in [AK70]. On every smooth compact connected manifold of dimension $m \geq 2$ admitting a non-trivial circle action $\mathcal{S} = \{S_t\}_{t \in \mathbb{S}^1}$ this method enables the construction of smooth diffeomorphisms with specific ergodic properties (e.g. weak mixing ones in [AK70], section 5, or [GK00]) or non-standard smooth realizations of measure preserving systems (e.g. [AK70], section 6, and [FSW07]). These diffeomorphisms are constructed as limits of conjugates $f_n = H_n \circ S_{\alpha_{n+1}} \circ H_n^{-1}$, where $\alpha_{n+1} = \alpha_n + \frac{1}{k_n \cdot l_n \cdot q_n^2} \in \mathbb{Q}$, $H_n = H_{n-1} \circ h_n$ and h_n is a measure-preserving diffeomorphism satisfying $R_{\frac{1}{q_n}} \circ h_n = h_n \circ R_{\frac{1}{q_n}}$. In each step the conjugation map h_n and the parameter k_n are chosen such that the diffeomorphism f_n imitates the desired property with a certain precision. Then the parameter l_n is chosen large enough to guarantee closeness of f_n to f_{n-1} in the C^∞ -topology and so the convergence of the sequence $(f_n)_{n \in \mathbb{N}}$ to a limit diffeomorphism is provided. See [FK04] for more details and other results of this method.

As a starting point we use the construction of weak mixing diffeomorphisms on \mathbb{T}^2 undertaken in the real-analytic topology in [FS05] with the explicit conjugation maps

$$\begin{aligned}\phi_n(\theta, r) &= (\theta, r + q_n^2 \cdot \cos(2\pi q_n \theta)), \\ g_n(\theta, r) &= (\theta + [nq_n^\sigma] \cdot r, r) \text{ with some } 0 < \sigma < \frac{1}{2}, \\ h_n &= g_n \circ \phi_n.\end{aligned}$$

Here $[\cdot]$ denotes the integer part of the number. Furthermore, let $\mathcal{R} = \{R_t\}_{t \in \mathbb{S}^1}$ denote the standard circle action on \mathbb{T}^2 comprising of the diffeomorphisms $R_t(\theta, r) = (\theta + t, r)$. Note that $h_n \circ R_{\frac{p_n}{q_n}} = R_{\frac{p_n}{q_n}} \circ h_n$. With the conjugation maps $H_n := h_1 \circ \dots \circ h_n$ we will define the diffeomorphism $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$. The sequence of rational numbers will be

$$\alpha_{n+1} = \frac{p_{n+1}}{q_{n+1}} = \alpha_n - \frac{a_n}{q_n \cdot \tilde{q}_{n+1}},$$

where $a_n \in \mathbb{Z}$, $1 \leq a_n \leq q_n$ is chosen in such a way that $\tilde{q}_{n+1} \cdot p_n \equiv a_n \pmod{q_n}$. Therewith, we have $|\alpha_{n+1} - \alpha_n| \leq \frac{1}{\tilde{q}_{n+1}}$ and $\tilde{q}_{n+1} \cdot \alpha_{n+1} = \frac{\tilde{q}_{n+1} \cdot p_n}{q_n} - \frac{a_n}{q_n} \equiv 0 \pmod{1}$, which implies $f_n^{\tilde{q}_{n+1}} = \text{id}$. Hence, $(\tilde{q}_n)_{n \in \mathbb{N}}$ will be a rigidity sequence of $f = \lim_{n \rightarrow \infty} f_n$ under some restrictions on the closeness between f_n and f (see Remark 4.6), which depend on the norms of the conjugation maps H_i and the distances $|\alpha_{i+1} - \alpha_i| \leq \frac{1}{\tilde{q}_{i+1}}$ for every $i > n$. Thus, we have to estimate the norms $\|H_n\|_n$ carefully. This will yield the subsequent requirement on the number \tilde{q}_{n+1} (see the end of section 4.2):

$$\tilde{q}_{n+1} > \varphi_1(n) \cdot \tilde{q}_n^{2 \cdot ((n+2) \cdot (n+1)^{n+1} + 1)},$$

where $\varphi_1(n) := 2^n \cdot (n+1)! \cdot ((n+2)!)^{(n+2)^{n-2} \cdot (n+1)} \cdot (2\pi n)^{(n+2) \cdot (n+1)^{n+1}}$. This is a sufficient condition on the growth rate of the rigidity sequence $(\tilde{q}_n)_{n \in \mathbb{N}}$ and we prove that f is weak mixing using a criterion similar to that deduced in [FS05] (see section 6). Consequently we obtain:

Theorem 1. *Let $\varphi_1(n) := 2^n \cdot (n+1)! \cdot ((n+2)!)^{(n+2)^{n-2} \cdot (n+1)} \cdot (2\pi n)^{(n+2) \cdot (n+1)^{n+1}}$. If $(\tilde{q}_n)_{n \in \mathbb{N}}$ is a sequence of natural numbers satisfying*

$$\tilde{q}_{n+1} \geq \varphi_1(n) \cdot \tilde{q}_n^{2 \cdot ((n+2) \cdot (n+1)^{n+1} + 1)},$$

there exists a weak mixing C^∞ -diffeomorphism of \mathbb{T}^2 that is uniformly rigid with respect to $(\tilde{q}_n)_{n \in \mathbb{N}}$.

In section 7.1 we conclude a rougher but more handsome statement:

Corollary 1. *If $(\tilde{q}_n)_{n \in \mathbb{N}}$ is a sequence of natural numbers satisfying $\tilde{q}_1 \geq 108\pi$ and $\tilde{q}_{n+1} \geq \tilde{q}_n^{\tilde{q}_n}$, then there exists a weak mixing C^∞ -diffeomorphism of \mathbb{T}^2 that is uniformly rigid with respect to $(\tilde{q}_n)_{n \in \mathbb{N}}$.*

We note that our requirement on the growth rate is less restrictive than the mentioned condition in [Ya13], Theorem 1.5.. In fact, the proof in [Ya13] shows that a condition of the form $\frac{n_{m+1}}{n_m} \geq n_m^{4n_m^2+20}$ is sufficient for her construction of a weakly mixing homeomorphism, which is uniformly rigid along $(n_m)_{m \in \mathbb{N}}$. Our requirement on the growth rate is still weaker.

By the same approach we consider the problem in the real-analytic topology. In this setting we will deduce the following sufficient condition on the growth rate of the rigidity sequence $(\tilde{q}_n)_{n \in \mathbb{N}}$:

Theorem 2. *Let $\rho > 0$. If $(\tilde{q}_n)_{n \in \mathbb{N}}$ is a sequence of natural numbers satisfying $\tilde{q}_1 \geq \rho + 1$ and*

$$\tilde{q}_{n+1} \geq 2^n \cdot 64\pi^2 \cdot n^2 \cdot \tilde{q}_n^{14} \cdot \exp(4\pi \cdot n \cdot \tilde{q}_n^6 \cdot \exp(2\pi \cdot \tilde{q}_n^4 \cdot (1 + n \cdot \tilde{q}_n))),$$

there exists a weak mixing Diff_ρ^ω -diffeomorphism of \mathbb{T}^2 that is uniformly rigid along the sequence $(\tilde{q}_n)_{n \in \mathbb{N}}$.

Again, we derive from this a more convenient statement in section 7.2:

Corollary 2. *If $(\tilde{q}_n)_{n \in \mathbb{N}}$ is a sequence of natural numbers satisfying $\tilde{q}_1 \geq (\rho + 1) \cdot 2^7 \cdot \pi^2$ and $\tilde{q}_{n+1} \geq \tilde{q}_n^{15} \cdot \exp(\tilde{q}_n^7 \cdot \exp(\tilde{q}_n^6))$, then there exists a weak mixing Diff_ρ^ω -diffeomorphism of \mathbb{T}^2 that is uniformly rigid with respect to $(\tilde{q}_n)_{n \in \mathbb{N}}$.*

2 Definitions and notations

In this chapter we want to introduce advantageous definitions and notations. In particular, we discuss topologies on the space of diffeomorphisms on \mathbb{T}^2 .

2.1 C^∞ -topology

For defining explicit metrics on $\text{Diff}^k(\mathbb{T}^2)$ and in the following the subsequent notations will be useful:

Definition 2.1. 1. For a sufficiently differentiable function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ and a multiindex $\vec{a} = (a_1, a_2) \in \mathbb{N}_0^2$

$$D_{\vec{a}}f := \frac{\partial^{|\vec{a}|}}{\partial x_1^{a_1} \partial x_2^{a_2}} f,$$

where $|\vec{a}| = a_1 + a_2$ is the order of \vec{a} .

2. For a continuous function $F : (0, 1)^2 \rightarrow \mathbb{R}$

$$\|F\|_0 := \sup_{z \in (0, 1)^2} |F(z)|.$$

For $f, g \in \text{Diff}^k(\mathbb{T}^2)$ let $F, G : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ denote their lifts. Furthermore, for a function $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ we denote by $[F]_i$ the i -th coordinate function.

Definition 2.2. 1. For $f, g \in \text{Diff}^k(\mathbb{T}^2)$ we define

$$\tilde{d}_0(f, g) = \max_{i=1,2} \left\{ \inf_{p \in \mathbb{Z}} \|[F - G]_i + p\|_0 \right\}$$

as well as

$$\tilde{d}_k(f, g) = \max \left\{ \tilde{d}_0(f, g), \|D_{\vec{a}}[F - G]_i\|_0 : i = 1, 2, 1 \leq |\vec{a}| \leq k \right\}.$$

2. Using the definitions from 1. we define for $f, g \in \text{Diff}^k(\mathbb{T}^2)$:

$$d_k(f, g) = \max \left\{ \tilde{d}_k(f, g), \tilde{d}_k(f^{-1}, g^{-1}) \right\}.$$

Obviously d_k describes a metric on $\text{Diff}^k(\mathbb{T}^2)$ measuring the distance between the diffeomorphisms as well as their inverses. As in the case of a general compact manifold the following definition connects to it:

Definition 2.3. 1. A sequence of $\text{Diff}^\infty(\mathbb{T}^2)$ -diffeomorphisms is called convergent in $\text{Diff}^\infty(\mathbb{T}^2)$ if it converges in $\text{Diff}^k(\mathbb{T}^2)$ for every $k \in \mathbb{N}$.

2. On $\text{Diff}^\infty(\mathbb{T}^2)$ we declare the following metric

$$d_\infty(f, g) = \sum_{k=1}^{\infty} \frac{d_k(f, g)}{2^k \cdot (1 + d_k(f, g))}.$$

It is a general fact that $\text{Diff}^\infty(\mathbb{T}^2)$ is a complete metric space with respect to this metric d_∞ . Moreover, we add the adjacent notation:

Definition 2.4. Let $f \in \text{Diff}^k(\mathbb{T}^2)$ with lift F be given. Then

$$\|Df\|_0 := \max_{i,j \in \{1,2\}} \|D_j[F]_i\|_0$$

and

$$|||f|||_k := \max \left\{ \|D_{\vec{a}}[F]_i\|_0, \|D_{\vec{a}}([F^{-1}]_i)\|_0 : i = 1, 2, \vec{a} \text{ multiindex with } 0 \leq |\vec{a}| \leq k \right\}.$$

2.2 Analytic topology

Real-analytic diffeomorphisms of \mathbb{T}^2 homotopic to the identity have a lift of type

$$F(\theta, r) = (\theta + f_1(\theta, r), r + f_2(\theta, r)),$$

where the functions $f_i : \mathbb{R}^2 \rightarrow \mathbb{R}$ are real-analytic and \mathbb{Z}^2 -periodic for $i = 1, 2$. For these functions we introduce the subsequent definition:

Definition 2.5. For any $\rho > 0$ we consider the set of real-analytic \mathbb{Z}^2 -periodic functions on \mathbb{R}^2 , that can be extended to a holomorphic function on $A^\rho := \{(\theta, r) \in \mathbb{C}^2 : |\text{Im}\theta| < \rho, |\text{Im}r| < \rho\}$.

1. For these functions let $\|f\|_\rho := \sup_{(\theta, r) \in A^\rho} |f(\theta, r)|$.

2. The set of these functions satisfying the condition $\|f\|_\rho < \infty$ is denoted by $C_\rho^\omega(\mathbb{T}^2)$.

Furthermore, we consider the space $\text{Diff}_\rho^\omega(\mathbb{T}^2)$ of those diffeomorphisms homotopic to the identity, for whose lift we have $f_i \in C_\rho^\omega(\mathbb{T}^2)$ for $i = 1, 2$.

Definition 2.6. For $f, g \in \text{Diff}_\rho^\omega(\mathbb{T}^2)$ we define

$$\|f\|_\rho = \max_{i=1,2} \|f_i\|_\rho$$

and the distance

$$d_\rho(f, g) = \max_{i=1,2} \left\{ \inf_{p \in \mathbb{Z}} \|f_i - g_i - p\|_\rho \right\}.$$

Remark 2.7. $\text{Diff}_\rho^\omega(\mathbb{T}^2)$ is a Banach space (see [Sa03] or [Ly99] for a more extensive treatment of these spaces).

Moreover, for a diffeomorphism T with lift $\tilde{T}(\theta, r) = (T_1(\theta, r), T_2(\theta, r))$ we define

$$\|DT\|_\rho = \max \left\{ \left\| \frac{\partial T_1}{\partial \theta} \right\|_\rho, \left\| \frac{\partial T_1}{\partial r} \right\|_\rho, \left\| \frac{\partial T_2}{\partial \theta} \right\|_\rho, \left\| \frac{\partial T_2}{\partial r} \right\|_\rho \right\}$$

and use the advantageous notation

$$\|T\|_\rho = \max \left\{ \inf_{k \in \mathbb{Z}} \sup_{(\theta, r) \in A_\rho} |T_1(\theta, r) - \theta + k|, \inf_{k \in \mathbb{Z}} \sup_{(\theta, r) \in A_\rho} |T_2(\theta, r) - r + k| \right\}.$$

3 Criterion for weak mixing

In this section we will formulate a criterion for weak mixing that will be used in the smooth as well as in the real-analytic case.

3.1 $(\gamma, \delta, \varepsilon)$ -distribution of horizontal intervals

Since we work on the manifold \mathbb{T}^2 , we recall the following definitions stated in [FS05]:

Definition 3.1. Let $\hat{\eta}$ be a partial decomposition of \mathbb{T} into intervals and consider on \mathbb{T}^2 the decomposition η consisting of intervals in $\hat{\eta}$ times some $r \in [0, 1]$. Sets of this form will be called horizontal intervals and decompositions of this type standard partial decompositions. On the other hand, sets of the form $\{\theta\} \times J$, where J is an interval on the r -axis, are called vertical intervals.

Hereby, we can introduce the notion of $(\gamma, \delta, \varepsilon)$ -distribution of a horizontal interval in the vertical direction:

Definition 3.2. A diffeomorphism $\Phi : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ $(\gamma, \delta, \varepsilon)$ -distributes a horizontal interval I if the following conditions are satisfied

- $\pi_r(\Phi(I))$ is an interval J with $1 - \delta \leq \lambda(J) \leq 1$,
- $\Phi(I)$ is contained in a vertical strip $[c, c + \gamma] \times J$ for some $c \in \mathbb{T}$,
- for any interval $\tilde{J} \subseteq J$ we have

$$\left| \frac{\lambda(I \cap \Phi^{-1}(\mathbb{T} \times \tilde{J}))}{\lambda(I)} - \frac{\lambda(\tilde{J})}{\lambda(J)} \right| \leq \varepsilon \cdot \frac{\lambda(\tilde{J})}{\lambda(J)}.$$

3.2 Statement of the criterion

The proof of the criterion is the same as in [FS05], section 3. The only difference occurs in comparison to Lemma 3.5., which in our case will be stated in the subsequent way:

Lemma 3.3. *Let $(\eta_n)_{n \in \mathbb{N}}$ be a sequence of standard partial decompositions of \mathbb{T}^2 into horizontal intervals of length less than $q_n^{-2.5}$. Moreover, let g_n be defined by $g_n(\theta, r) = (\theta + [nq_n^\sigma] \cdot r, r)$ with some $0 < \sigma < 1$ and let $(H_n)_{n \in \mathbb{N}}$ be a sequence of area-preserving diffeomorphisms such that for every $n \in \mathbb{N}$:*

$$(C1) \quad \|DH_{n-1}\|_0 \leq q_n^{0.5}.$$

Consider the partitions $\nu_n := \{\Gamma_n = H_{n-1}(g_n(I_n)) : I_n \in \eta_n\}$. Then $\eta_n \rightarrow \epsilon$ implies $\nu_n \rightarrow \epsilon$.

Proof. For every $\varepsilon > 0$ we can choose n large enough such that $\mu\left(\bigcup_{I \in \eta_n} I\right) > 1 - \varepsilon$ (because of $\eta_n \rightarrow \epsilon$) and there is a collection of squares $\tilde{S}_n := \{S_{n,i}\}$ with side length between $q_n^{-1.5}$ and q_n^{-2} with total measure of the union $S_n := \bigcup_i S_{n,i}$ greater than $1 - \sqrt{\varepsilon}$. Then we have $\mu\left(\bigcup_{I \in \eta_n} I \cap S_n\right) \geq (1 - \sqrt{\varepsilon}) \cdot \mu(S_n)$, because otherwise $\mu\left(S_n \setminus \bigcup_{I \in \eta_n} I\right) > \sqrt{\varepsilon} \cdot \mu(S_n) > \sqrt{\varepsilon} \cdot (1 - \sqrt{\varepsilon})$ and so $\mu\left(\mathbb{T}^2 \setminus \bigcup_{I \in \eta_n} I\right) > \sqrt{\varepsilon} - \varepsilon > \varepsilon$ in case of $\varepsilon < \frac{1}{4}$, which contradicts $\mu\left(\bigcup_{I \in \eta_n} I\right) > 1 - \varepsilon$. Since the horizontal intervals $I \in \eta_n$ have length less than $q_n^{-2.5}$, we can approximate the squares in the above collection \tilde{S}_n for n sufficiently large in such a way that $\mu\left(\bigcup_{I \in \eta_n, I \subset S_n} I\right) \geq (1 - 2\sqrt{\varepsilon}) \cdot \mu(S_n)$.

In the next step we consider the sets $C_{n,i} := H_{n-1}(g_n(S_{n,i}))$ with $S_{n,i} \in \tilde{S}_n$. For these sets $C_{n,i}$ we have:

$$\text{diam}(C_{n,i}) \leq \|DH_{n-1}\|_0 \cdot \|Dg_n\|_0 \cdot \text{diam}(S_{n,i}) \leq q_n^{0.5} \cdot n \cdot q_n^\sigma \cdot \sqrt{2} \cdot q_n^{-1.5} = n \cdot \sqrt{2} \cdot q_n^{\sigma-1},$$

which goes to 0 as $n \rightarrow \infty$ because $\sigma < 1$. Therefore, any Borel set B can be approximated by a union of such sets $C_{n,i}$ with any prescribed accuracy if n is sufficiently large, i.e. for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for $n \geq N$ there is an index set J_n : $\mu(B \triangle \bigcup_{i \in J_n} C_{n,i}) < \varepsilon$. Now we choose the union of these elements $I \in \eta_n$ contained in the occurring cubes $S_{n,i}$ and obtain: $\mu(B \triangle \bigcup H_{n-1} \circ g_n(I)) \leq \mu(B \triangle \bigcup_{i \in J_n} C_{n,i}) + \mu\left(S_n \setminus \bigcup_{I \in \eta_n, I \subset S_n} I\right) < \varepsilon + 2\sqrt{\varepsilon} \cdot \mu(S_n) < 3\sqrt{\varepsilon}$. Thus, B gets well approximated by unions of elements of ν_n if n is chosen sufficiently large. \square

Now the criterion for weak mixing can be stated in the following way (compare with [FS05], Proposition 3.9.):

Proposition 3.4. *Let $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$ be diffeomorphisms constructed as explained in the introduction with $0 < \sigma < \frac{1}{2}$ and such that $\|DH_{n-1}\|_0 \leq q_n^{0.5}$ holds for all $n \in \mathbb{N}$. Suppose that the limit $f := \lim_{n \rightarrow \infty} f_n$ exists. If there exists a sequence $(m_n)_{n \in \mathbb{N}}$ of natural numbers satisfying $d_0(f_n^{m_n}, f^{m_n}) < \frac{1}{2^n}$ and a sequence $(\eta_n)_{n \in \mathbb{N}}$ of standard partial decompositions of \mathbb{T}^2 into horizontal intervals of length less than $q_n^{-2.5}$ such that $\eta_n \rightarrow \epsilon$ and the diffeomorphism $\Phi_n := \phi_n \circ R_{\alpha_{n+1}}^{m_n} \circ \phi_n^{-1} \left(\frac{1}{nq_n^\sigma}, \frac{1}{n}, \frac{1}{n} \right)$ -distributes every interval $I_n \in \eta_n$, then the limit diffeomorphism f is weak mixing.*

Remark 3.5. In [FS05] it is demanded $\|DH_{n-1}\|_0 < \ln(q_n)$ instead of requirement C1. We did this modification because the fulfilment of the original condition would lead to stricter requirements on the rigidity sequence: In particular, equation A3 would require an exponential growth rate.

4 Convergence of $(f_n)_{n \in \mathbb{N}}$ in $\text{Diff}^\infty(\mathbb{T}^2)$

4.1 Properties of the conjugation maps h_n and H_n

Using the explicit definitions of the maps g_n, ϕ_n we can compute

$$h_n(\theta, r) = (\theta + [nq_n^\sigma] \cdot r + [nq_n^\sigma] \cdot q_n^2 \cdot \cos(2\pi q_n \theta), r + q_n^2 \cdot \cos(2\pi q_n \theta))$$

as well as

$$h_n^{-1}(\theta, r) = (\theta - [nq_n^\sigma] \cdot r, r - q_n^2 \cdot \cos(2\pi q_n (\theta - [nq_n^\sigma] \cdot r))).$$

Then we can estimate for every $k \in \mathbb{N}$ and every multiindex $\vec{a} \in \mathbb{N}_0^2$, $|\vec{a}| \leq k$:

$$\|D_{\vec{a}} h_n\|_0 \leq 2 \cdot (2\pi)^k \cdot [nq_n^\sigma] \cdot q_n^{2+k}$$

and

$$\|D_{\vec{a}} h_n^{-1}\|_0 \leq 2 \cdot (2\pi)^k \cdot [nq_n^\sigma]^k \cdot q_n^{2+k}.$$

Thus, we obtain

$$(1) \quad \|h_n\|_k \leq 2^{k+1} \cdot \pi^k \cdot q_n^{2+k} \cdot n^k \cdot q_n^{\sigma \cdot k} \leq (2\pi n q_n^2)^{k+1}.$$

In the next step we want to deduce norm estimates for the conjugation map $H_n = H_{n-1} \circ h_n$. Therefore, we have to understand the derivatives of a composition of maps:

Lemma 4.1. *Let $g, h \in \text{Diff}^\infty(\mathbb{T}^2)$ and $k \in \mathbb{N}$. Then for the composition $g \circ h$ it holds*

$$\|g \circ h\|_k \leq (k+1)! \cdot \|g\|_k^k \cdot \|h\|_k^k.$$

Proof. By induction on $k \in \mathbb{N}$ we will prove the following observation:

Claim: *For any multiindex $\vec{a} \in \mathbb{N}_0^2$ with $|\vec{a}| = k$ and $i \in \{1, 2\}$ the partial derivative $D_{\vec{a}}[g \circ h]_i$ consists of at most $(k+1)!$ summands, where each summand is the product of one derivative of g of order at most k and at most k derivatives of h of order at most k .*

- *Start:* $k = 1$
For $i_1, i \in \{1, 2\}$ we compute:

$$D_{x_{i_1}}[g \circ h]_i(x_1, x_2) = \sum_{j_1=1}^2 (D_{x_{j_1}}[g]_i)(h(x_1, x_2)) \cdot D_{x_{i_1}}[h]_{j_1}(x_1, x_2).$$

Hence, this derivative consists of $2! = 2$ summands and each summand has the announced form.

- *Induction assumption:* The claim holds for $k \in \mathbb{N}$.
- *Induction step:* $k \rightarrow k+1$

Let $i \in \{1, 2\}$ and $\vec{b} \in \mathbb{N}_0^2$ be any multiindex of order $|\vec{b}| = k+1$. There are $j \in \{1, 2\}$ and a multiindex \vec{a} of order $|\vec{a}| = k$ such that $D_{\vec{b}} = D_{x_j} D_{\vec{a}}$. By the induction assumption the partial derivative $D_{\vec{a}}[g \circ h]_i$ consists of at most $(k+1)!$ summands, at which the summand with the most factors is of the subsequent form:

$$D_{\vec{c}_1}[g]_i(h(x_1, x_2)) \cdot D_{\vec{c}_2}[h]_{i_2}(x_1, x_2) \cdot \dots \cdot D_{\vec{c}_{k+1}}[h]_{i_{k+1}}(x_1, x_2),$$

where each \vec{c}_i is of order at most k . Using the product rule we compute how the derivative D_{x_j} acts on such a summand:

$$\left(\sum_{j_1=1}^2 D_{x_{j_1}} D_{\vec{c}_1} [g]_i \circ h \cdot D_{x_j} [h]_{j_1} D_{\vec{c}_2} [h]_{i_2} \cdot \dots \cdot D_{\vec{c}_{k+1}} [h]_{i_{k+1}} \right) + \\ D_{\vec{c}_1} [g]_i \circ h \cdot D_{x_j} D_{\vec{c}_2} [h]_{i_2} \cdot \dots \cdot D_{\vec{c}_{k+1}} [h]_{i_{k+1}} + \dots + D_{\vec{c}_1} [g]_i \circ h \cdot D_{\vec{c}_2} [h]_{i_2} \cdot \dots \cdot D_{x_j} D_{\vec{c}_{k+1}} [h]_{i_{k+1}}$$

Thus, each summand is the product of one derivative of g of order at most $k+1$ and at most $k+1$ derivatives of h of order at most $k+1$. Moreover, we observe that $2+k$ summands arise out of one. So the number of summands can be estimated by $(k+2) \cdot (k+1)! = (k+2)!$ and the claim is verified.

Using this claim we obtain for $i \in \{1, 2\}$ and any multiindex $\vec{a} \in \mathbb{N}_0^2$ of order $|\vec{a}| = k$:

$$\|D_{\vec{a}} [g \circ h]_i\|_0 \leq (k+1)! \cdot \|g\|_k \cdot \|h\|_k^k.$$

Applying the claim on $h^{-1} \circ g^{-1}$ yields:

$$\|D_{\vec{a}} [h^{-1} \circ g^{-1}]_i\|_0 \leq (k+1)! \cdot \|g\|_k^k \cdot \|h\|_k.$$

We conclude

$$\|g \circ h\|_k \leq (k+1)! \cdot \|g\|_k^k \cdot \|h\|_k^k.$$

□

Using this result we compute for every $k \in \mathbb{N}$:

$$(2) \quad \|H_n\|_k \leq (k+1)! \cdot \|H_{n-1}\|_k^k \cdot \|h_n\|_k^k.$$

Hereby, we can deduce the subsequent estimate of the norm $\|H_n\|_{k+1}$ under some assumptions on the growth rate of the numbers q_n :

Lemma 4.2. *Let $k, n \in \mathbb{N}$ and $n \geq 2$. Assume*

$$(A2) \quad q_{n+1} \geq 2 \cdot \pi \cdot n \cdot q_n^2.$$

Then we have

$$\|H_n\|_{k+1} \leq ((k+2)!)^{(k+2)^{n-2}} \cdot (2\pi n q_n)^{(k+2) \cdot (k+1)^{n-1} \cdot (n+1)}.$$

Proof. Let $k \in \mathbb{N}$ be arbitrary. We proof this result by induction on n :

- *Start:* $n = 2$

Using equation 2 and the norm estimate on h_n from equation 1 we obtain the claim:

$$\begin{aligned} \|H_2\|_{k+1} &\leq (k+2)! \cdot \|H_1\|_{k+1}^{k+1} \cdot \|h_2\|_{k+1}^{k+1} \\ &= (k+2)! \cdot \|h_1\|_{k+1}^{k+1} \cdot \|h_2\|_{k+1}^{k+1} \\ &\leq (k+2)! \cdot \left((2\pi q_1^2)^{k+2} \right)^{k+1} \cdot \left((2\pi \cdot 2 \cdot q_2^2)^{k+2} \right)^{k+1} \\ &\leq (k+2)! \cdot q_2^{(k+2) \cdot (k+1)} \cdot (2\pi \cdot 2 \cdot q_2)^{2 \cdot (k+2) \cdot (k+1)} \\ &\leq (k+2)! \cdot (2\pi \cdot 2 \cdot q_2)^{3 \cdot (k+2) \cdot (k+1)} \\ &= ((k+2)!)^{(k+2)^{2-2}} \cdot (2\pi \cdot 2 \cdot q_2)^{(k+2) \cdot (k+1)^{2-1} \cdot (2+1)} \end{aligned}$$

• *Induction assumption:* The claim is true for $n \in \mathbb{N}$, $n \geq 2$.

• *Induction step $n \rightarrow n+1$:*

Using equation 2, the norm estimate on h_n from equation 1 and the induction assumption we compute:

$$\begin{aligned} |||H_{n+1}|||_{k+1} &\leq (k+2)! \cdot |||H_n|||_{k+1}^{k+1} \cdot |||h_{n+1}|||_{k+1}^{k+1} \\ &\leq (k+2)! \cdot \left(((k+2)!)^{(k+2)^{n-2}} \cdot (2\pi n q_n)^{(k+2) \cdot (k+1)^{n-1} \cdot (n+1)} \right)^{k+1} \cdot \left((2\pi(n+1) q_{n+1}^2)^{k+2} \right)^{k+1} \\ &\leq (k+2)! \cdot ((k+2)!)^{(k+2)^{n-2} \cdot (k+1)} \cdot q_{n+1}^{(k+2) \cdot (k+1)^n \cdot (n+1)} \cdot (2\pi(n+1) q_{n+1})^{2 \cdot (k+2) \cdot (k+1)} \\ &\leq ((k+2)!)^{(k+2)^{n-1}} \cdot (2\pi(n+1) q_{n+1})^{(k+2) \cdot (k+1)^n \cdot (n+2)}, \end{aligned}$$

where we used in the last step the subsequent estimation:

$$\begin{aligned} (k+2) \cdot (k+1)^n \cdot (n+1) + 2 \cdot (k+2) \cdot (k+1) &= (k+2) \cdot (k+1) \cdot \left((k+1)^{n-1} \cdot (n+1) + 2 \right) \\ &\leq (k+2) \cdot (k+1) \cdot (k+1)^{n-1} \cdot (n+2) = (k+2) \cdot (k+1)^n \cdot (n+2). \end{aligned}$$

□

Remark 4.3. As a special case of Lemma 4.1 we have $\|DH_n\|_0 \leq 2! \cdot \|DH_{n-1}\|_0 \cdot \|Dh_n\|_0$. With the aid of equation 1 we can estimate:

$$\|DH_n\|_0 \leq 2! \cdot q_n^{0.5} \cdot (2\pi \cdot n \cdot q_n^2)^2 = 8\pi^2 \cdot n^2 \cdot q_n^{4.5},$$

where we used condition C1, i.e. $\|DH_{n-1}\|_0 \leq q_n^{0.5}$. In order to guarantee this property for DH_n we demand:

$$(A3) \quad q_{n+1} \geq \|DH_n\|_0^2 \geq (8\pi^2 \cdot n^2 \cdot q_n^{4.5})^2 = 64\pi^4 \cdot n^4 \cdot q_n^9.$$

4.2 Proof of Convergence

In the proof of convergence the following result, which is more precise than [FS05], Lemma 5.6., is useful:

Lemma 4.4. *Let $k \in \mathbb{N}_0$ and $h \in \text{Diff}^\infty(\mathbb{T}^2)$. Then for all $\alpha, \beta \in \mathbb{R}$ we obtain:*

$$d_k(h \circ R_\alpha \circ h^{-1}, h \circ R_\beta \circ h^{-1}) \leq C_k \cdot |||h|||_{k+1}^{k+1} \cdot |\alpha - \beta|,$$

where $C_k = (k+1)!$.

Proof. As an application of the claim in the proof of Lemma 4.1 we observe

Fact: For any $\vec{a} \in \mathbb{N}_0^2$ with $|\vec{a}| = k$ and $i \in \{1, 2\}$ the partial derivative $D_{\vec{a}}[h \circ R_\alpha \circ h^{-1}]_i$ consists of at most $(k+1)!$ summands, where each summand is the product of one derivative of h of order at most k and at most k derivatives of h^{-1} of order at most k .

Furthermore, with the aid of the mean value theorem we can estimate for any multiindex $\vec{a} \in \mathbb{N}_0^2$ with $|\vec{a}| \leq k$ and $i \in \{1, 2\}$:

$$|D_{\vec{a}}[h]_i(R_\alpha \circ h^{-1}(x_1, x_2)) - D_{\vec{a}}[h]_i(R_\beta \circ h^{-1}(x_1, x_2))| \leq |||h|||_{k+1} \cdot |\alpha - \beta|.$$

Since $(h_n \circ R_\alpha \circ h_n^{-1})^{-1} = h_n \circ R_{-\alpha} \circ h_n^{-1}$ is of the same form, we obtain in conclusion:

$$\begin{aligned} d_k(h \circ R_\alpha \circ h^{-1}, h \circ R_\beta \circ h^{-1}) &\leq (k+1)! \cdot |||h|||_{k+1} \cdot |||h|||_k^k \cdot |\alpha - \beta| \\ &\leq (k+1)! \cdot |||h|||_{k+1}^{k+1} \cdot |\alpha - \beta|. \end{aligned}$$

□

Under some conditions on the proximity of α_n and α_{n+1} we can prove convergence:

Lemma 4.5. *We assume*

$$(A1) \quad |\alpha_{n+1} - \alpha_n| \leq \frac{1}{2^n \cdot (n+1)! \cdot q_n \cdot |||H_n|||_{n+1}^{n+1}}.$$

Then the diffeomorphisms $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$ satisfy:

- The sequence $(f_n)_{n \in \mathbb{N}}$ converges in the $\text{Diff}^\infty(\mathbb{T}^2)$ -topology to a measure-preserving diffeomorphism f .
- We have for every $n \in \mathbb{N}$ and $m \leq q_{n+1}$:

$$d_0(f^m, f_n^m) < \frac{1}{2^n}.$$

Proof. 1. According to our construction it holds $h_n \circ R_{\alpha_n} = R_{\alpha_n} \circ h_n$ and hence we can apply Lemma 4.4 for every $k, n \in \mathbb{N}$:

$$d_k(f_n, f_{n-1}) = d_k(H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}, H_n \circ R_{\alpha_n} \circ H_n^{-1}) \leq C_k \cdot |||H_n|||_{k+1}^{k+1} \cdot |\alpha_{n+1} - \alpha_n|.$$

By the assumptions of this Lemma it follows for every $k \leq n$:

$$(3) \quad d_k(f_n, f_{n-1}) \leq d_n(f_n, f_{n-1}) \leq C_n \cdot |||H_n|||_{n+1}^{n+1} \cdot \frac{1}{2^n \cdot C_n \cdot q_n \cdot |||H_n|||_{n+1}^{n+1}} < \frac{1}{2^n}.$$

In the next step we show that for arbitrary $k \in \mathbb{N}$ $(f_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $\text{Diff}^k(\mathbb{T}^2)$, i.e. $\lim_{n, m \rightarrow \infty} d_k(f_n, f_m) = 0$. For this purpose, we calculate:

$$(4) \quad \lim_{n \rightarrow \infty} d_k(f_n, f_m) \leq \lim_{n \rightarrow \infty} \sum_{i=m+1}^n d_k(f_i, f_{i-1}) = \sum_{i=m+1}^{\infty} d_k(f_i, f_{i-1}).$$

We consider the limit process $m \rightarrow \infty$, i.e. we can assume $k \leq m$ and obtain from equations 3 and 4:

$$\lim_{n, m \rightarrow \infty} d_k(f_n, f_m) \leq \lim_{m \rightarrow \infty} \sum_{i=m+1}^{\infty} \frac{1}{2^i} = 0.$$

Since $\text{Diff}^k(\mathbb{T}^2)$ is complete, the sequence $(f_n)_{n \in \mathbb{N}}$ converges consequently in $\text{Diff}^k(\mathbb{T}^2)$ for every $k \in \mathbb{N}$. Thus, the sequence converges in $\text{Diff}^\infty(\mathbb{T}^2)$ by definition.

2. Again with the help of Lemma 4.4 we compute for every $i \in \mathbb{N}$:

$$d_0(f_i^m, f_{i-1}^m) = d_0(H_i \circ R_{m \cdot \alpha_{i+1}} \circ H_i^{-1}, H_i \circ R_{m \cdot \alpha_i} \circ H_i^{-1}) \leq |||H_i|||_1 \cdot m \cdot |\alpha_{i+1} - \alpha_i|.$$

Since $m \leq q_{n+1} \leq q_i$ we conclude for every $i > n$:

$$d_0(f_i^m, f_{i-1}^m) \leq |||H_i|||_1 \cdot m \cdot \frac{1}{2^i \cdot (i+1)! \cdot q_i \cdot |||H_i|||_{i+1}^{i+1}} < \frac{m}{q_i} \cdot \frac{1}{2^i} \leq \frac{1}{2^i}.$$

Thus, for every $m \leq q_{n+1}$ we get the claimed result:

$$d_0(f^m, f_n^m) = \lim_{k \rightarrow \infty} d_0(f_k^m, f_n^m) \leq \lim_{k \rightarrow \infty} \sum_{i=n+1}^k d_0(f_i^m, f_{i-1}^m) < \sum_{i=n+1}^{\infty} \frac{1}{2^i} = \left(\frac{1}{2}\right)^n.$$

□

Remark 4.6. By definition $\tilde{q}_{n+1} \leq q_{n+1}$. Hence, the second statement of the previous Lemma implies $d_0 \left(f_n^{\tilde{q}_{n+1}}, f_{\tilde{q}_{n+1}} \right) < \frac{1}{2^n}$. Since the number α_{n+1} was chosen in such a way that $f_n^{\tilde{q}_{n+1}} = id$, we have $d_0(id, f_{\tilde{q}_{n+1}}) < \frac{1}{2^n}$, which goes to zero as $n \rightarrow \infty$. Thus, $(\tilde{q}_n)_{n \in \mathbb{N}}$ is an uniform rigidity sequence of f .

By Lemma 4.2 we can satisfy the requirement A1 if we demand:

$$|\alpha_{n+1} - \alpha_n| \leq \frac{1}{2^n \cdot (n+1)! \cdot q_n \cdot ((n+2)!)^{(n+2)^{n-2} \cdot (n+1)} \cdot (2\pi n q_n)^{(n+2) \cdot (n+1)^{n+1}}}.$$

Since $|\alpha_{n+1} - \alpha_n| = \frac{a_n}{q_n \cdot \tilde{q}_{n+1}} \leq \frac{1}{\tilde{q}_{n+1}}$ this requirement can be met if we demand

$$\tilde{q}_{n+1} \geq \varphi_1(n) \cdot q_n^{(n+2) \cdot (n+1)^{n+1} + 1},$$

at which $\varphi_1(n) := 2^n \cdot (n+1)! \cdot ((n+2)!)^{(n+2)^{n-2} \cdot (n+1)} \cdot (2\pi n)^{(n+2) \cdot (n+1)^{n+1}}$. Hereby, the other conditions A3 and A2 are fulfilled.

Using $q_n = q_{n-1} \cdot \tilde{q}_n < \tilde{q}_n^2$ this yields the condition

$$\tilde{q}_{n+1} \geq \varphi_1(n) \cdot \tilde{q}_n^{2 \cdot ((n+2) \cdot (n+1)^{n+1} + 1)}.$$

5 Convergence of $(f_n)_{n \in \mathbb{N}}$ in $\text{Diff}_\rho^\omega(\mathbb{T}^2)$

Let $\rho > 0$ be given.

5.1 Properties of the conjugation maps h_n and H_n

Regarded as a function $h_n : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ we have

$$\begin{aligned} h_n^{-1}(\theta, r) &= h_n^{-1}(\theta_1 + \imath \cdot \theta_2, r_1 + \imath \cdot r_2) \\ &= (\theta_1 + \imath \cdot \theta_2 - [nq_n^\sigma] \cdot (r_1 + \imath \cdot r_2), r_1 + \imath \cdot r_2 - q_n^2 \cdot \cos(2\pi q_n(\theta_1 + \imath \cdot \theta_2 - [nq_n^\sigma] \cdot (r_1 + \imath \cdot r_2))))). \end{aligned}$$

Since for $(\theta_1 + \imath \cdot \theta_2, r_1 + \imath \cdot r_2) \in A_\rho$ it holds $-\rho < r_2 < \rho$ and $-\rho < \theta_2 < \rho$, we can estimate:

$$\begin{aligned} \inf_{k \in \mathbb{Z}} \sup_{(\theta, r) \in A_\rho} |[h_n^{-1}]_1 - \theta + k| &= \inf_{k \in \mathbb{Z}} \sqrt{(-[nq_n^\sigma] \cdot r_1 + k)^2 + (-[nq_n^\sigma] \cdot r_2)^2} \\ &\leq [nq_n^\sigma] \cdot \sqrt{1 + \rho^2} \end{aligned}$$

and

$$\inf_{k \in \mathbb{Z}} \sup_{(\theta, r) \in A_\rho} |[h_n^{-1}]_2 - r + k| \leq 2 \cdot q_n^2 \cdot \exp(2\pi q_n \cdot \rho + 2\pi q_n \cdot [nq_n^\sigma] \cdot \rho).$$

Hence, it holds (note that we demand $q_1 \geq \tilde{\rho}_0 = \rho + 1$ in equation B2'):

$$(5) \quad \|h_n^{-1}\|_\rho \leq 2 \cdot q_n^2 \cdot \exp(2\pi \cdot q_n \cdot \rho \cdot (1 + [nq_n^\sigma])).$$

We introduce the subsequent quantities:

$$\begin{aligned} \rho_n &:= \|H_n^{-1}\|_\rho, & \rho_0 &:= \rho \\ \tilde{\rho}_n &:= 2 \cdot q_n^2 \cdot \exp(2\pi \cdot q_n \cdot \tilde{\rho}_{n-1} \cdot (1 + [nq_n^\sigma])), & \tilde{\rho}_0 &:= \rho + 1 \end{aligned}$$

Using equation 5 we obtain

$$\rho_n = \|h_n^{-1} \circ H_{n-1}^{-1}\|_\rho \leq \|h_n^{-1}\|_{\rho_{n-1}} \leq 2 \cdot q_n^2 \cdot \exp(2\pi q_n \cdot \rho_{n-1} \cdot (1 + [nq_n^\sigma])).$$

We state the following relation between the quantities:

Remark 5.1. We have $\rho_n + 1 \leq \tilde{\rho}_n$ for every $n \in \mathbb{N}$.

Proof. We prove this result by induction on n :

- *Start:* $n = 1$

By the above formula we verify:

$$\begin{aligned} \tilde{\rho}_1 &= 2 \cdot q_1^2 \cdot \exp(2\pi \cdot q_1 \cdot \tilde{\rho}_0 \cdot (1 + [q_1^\sigma])) \\ &= 2 \cdot q_1^2 \cdot \exp(2\pi \cdot q_1 \cdot \rho \cdot (1 + [q_1^\sigma])) \cdot \exp(2\pi \cdot q_1 \cdot (1 + [q_1^\sigma])) \\ &\geq 2 \cdot q_1^2 \cdot \exp(2\pi \cdot q_1 \cdot \rho \cdot (1 + [q_1^\sigma])) + 1 \geq \rho_1 + 1. \end{aligned}$$

- *Induction assumption:* The claim holds for $n - 1$.

- *Induction step:* $n - 1 \rightarrow n$

Using the induction assumption we can compute in the same way:

$$\begin{aligned} \tilde{\rho}_n &= 2 \cdot q_n^2 \cdot \exp(2\pi \cdot q_n \cdot \tilde{\rho}_{n-1} \cdot (1 + [nq_n^\sigma])) \\ &\geq 2 \cdot q_n^2 \cdot \exp(2\pi \cdot q_n \cdot \rho_{n-1} \cdot (1 + [nq_n^\sigma])) \cdot \exp(2\pi \cdot q_n \cdot (1 + [nq_n^\sigma])) \\ &\geq 2 \cdot q_n^2 \cdot \exp(2\pi \cdot q_n \cdot \rho_{n-1} \cdot (1 + [nq_n^\sigma])) + 1 \geq \rho_n + 1. \end{aligned}$$

□

We demand

$$(B2') \quad q_{n+1} \geq \tilde{\rho}_n.$$

This yields the condition: $q_{n+1} \geq 2 \cdot q_n^2 \cdot \exp(2\pi q_n \cdot \tilde{\rho}_{n-1} \cdot (1 + [nq_n^\sigma]))$. And since we demand $\tilde{\rho}_{n-1} \leq q_n$ we require for the sequence $(q_n)_{n \in \mathbb{N}}$:

$$(B2) \quad q_{n+1} \geq 2 \cdot q_n^2 \cdot \exp(2\pi \cdot q_n^2 \cdot (1 + nq_n^\sigma)).$$

Furthermore, recall that

$$h_n(\theta, r) = (\theta + [nq_n^\sigma] \cdot r + [nq_n^\sigma] \cdot q_n^2 \cdot \cos(2\pi q_n \theta), r + q_n^2 \cdot \cos(2\pi q_n \theta)).$$

The occurring partial derivatives are

$$\begin{aligned} \frac{\partial [h_n]_1}{\partial \theta} &= 1 - [nq_n^\sigma] \cdot 2\pi \cdot q_n^3 \cdot \sin(2\pi q_n \theta) & \frac{\partial [h_n]_1}{\partial r} &= [nq_n^\sigma] \\ \frac{\partial [h_n]_2}{\partial \theta} &= -2\pi \cdot q_n^3 \cdot \sin(2\pi q_n \theta) & \frac{\partial [h_n]_2}{\partial r} &= 1 \end{aligned}$$

Thus, in order to calculate $\|Dh_n\|_\rho$, we have to examine $\left\| \frac{\partial [h_n]_1}{\partial \theta} \right\|_\rho$:

$$\|Dh_n\|_\rho = \left\| \frac{\partial [h_n]_1}{\partial \theta} \right\|_\rho \leq 1 + [nq_n^\sigma] \cdot 2\pi \cdot q_n^3 \cdot \exp(2\pi \cdot q_n \cdot \rho) \leq 4\pi \cdot n \cdot q_n^{3+\sigma} \cdot \exp(2\pi \cdot q_n \cdot \rho).$$

Under condition B2' we can estimate with the aid of Remark 5.1:

$$(6) \quad \begin{aligned} \|Dh_n\|_{\rho_{n+1}} &\leq 4\pi n \cdot q_n^{3+\sigma} \cdot \exp(2\pi \cdot q_n \cdot (\rho_n + 1)) \leq 4\pi n \cdot q_n^{3+\sigma} \cdot \exp(2\pi \cdot q_n \cdot \tilde{\rho}_n) \\ &\leq 4\pi \cdot n \cdot q_n^{3+\sigma} \cdot \exp(4\pi \cdot q_n^3 \cdot \exp(2\pi \cdot q_n^2 \cdot (1 + n \cdot q_n^\sigma))) \end{aligned}$$

In order to be able to apply the criterion for weak mixing 3.4 we have the requirement C1: $q_{n+1} \geq \|DH_n\|_0^2$. Using the above calculations we obtain

$$\|DH_n\|_0 \leq 2! \cdot \|DH_{n-1}\|_0 \cdot \|Dh_n\|_0 \leq 2 \cdot q_n^{0.5} \cdot 4\pi \cdot n \cdot q_n^{3+\sigma} \leq 8\pi \cdot n \cdot q_n^{3.5+\sigma}.$$

So we demand

$$(B3) \quad q_{n+1} \geq 64\pi^2 \cdot n^2 \cdot q_n^8.$$

5.2 Proof of Convergence

As a preparatory result we prove the subsequent Lemma:

Lemma 5.2. *Let $n \in \mathbb{N}$, $m \in \mathbb{N}_0$. Under the condition*

$$\left\| h_n \circ R_{\alpha_{n+1}}^m \circ h_n^{-1} - R_{\alpha_n}^m \right\|_{\rho_{n-1}} < \frac{1}{2^n \cdot \|Dh_1\|_{\rho_1+1} \cdot \dots \cdot \|Dh_{n-1}\|_{\rho_{n-1}+1}}$$

we have $d_\rho(f_n^m, f_{n-1}^m) < \frac{1}{2^n}$.

Proof. We introduce the functions $\psi_{n,k} := h_{k+1} \circ \dots \circ h_n \circ R_{\alpha_{n+1}}^m \circ h_n^{-1} \dots \circ h_{k+1}^{-1}$ in case of $1 \leq k \leq n-1$ and $\psi_{n,n} = R_{\alpha_{n+1}}^m$. With these we have $f_n^m = h_1 \circ \dots \circ h_k \circ \psi_{n,k} \circ h_k^{-1} \circ \dots \circ h_1^{-1}$. By induction on $k \in \mathbb{N}$ in the range $1 \leq k \leq n-1$ we prove:

Fact: Under the condition $\left\| \psi_{n,k} \circ H_k^{-1} - \psi_{n-1,k} \circ H_k^{-1} \right\|_\rho < \frac{1}{2^n \cdot \|Dh_1\|_{\rho_1+1} \cdot \dots \cdot \|Dh_k\|_{\rho_k+1}}$ we have

$$\left\| h_1 \circ \dots \circ h_k \circ \psi_{n,k} \circ h_k^{-1} \circ \dots \circ h_1^{-1} - h_1 \circ \dots \circ h_k \circ \psi_{n-1,k} \circ h_k^{-1} \circ \dots \circ h_1^{-1} \right\|_\rho < \frac{1}{2^n}.$$

- *Start:* $k = 1$

At first we note $h_1^{-1}(A^\rho) \subseteq A^{\rho_1}$. By our assumption we have

$$\left\| \psi_{n,1} \circ h_1^{-1} - \psi_{n-1,1} \circ h_1^{-1} \right\|_\rho < \frac{1}{2^n \cdot \|Dh_1\|_{\rho_1+1}} < 1.$$

So a sufficient condition for our claim is given by

$$\|Dh_1\|_{\rho_1+1} \cdot \left\| \psi_{n,1} \circ h_1^{-1} - \psi_{n-1,1} \circ h_1^{-1} \right\|_\rho < \frac{1}{2^n},$$

which is satisfied by our requirements.

- *Induction hypothesis:* The claim holds for $1 \leq k-1 \leq n-2$.

- *Induction step:* $k-1 \rightarrow k$

Using the induction hypothesis the proximity

$$\begin{aligned} \left\| \psi_{n,k-1} \circ H_{k-1}^{-1} - \psi_{n-1,k-1} \circ H_{k-1}^{-1} \right\|_\rho &= \left\| h_k \circ \psi_{n,k} \circ H_k^{-1} - h_k \circ \psi_{n-1,k} \circ H_k^{-1} \right\|_\rho \\ &< \frac{1}{2^n \cdot \|Dh_1\|_{\rho_1+1} \cdot \dots \cdot \|Dh_{k-1}\|_{\rho_{k-1}+1}} \end{aligned}$$

is sufficient to prove the claim. Since $\|\psi_{n,k} \circ H_k^{-1} - \psi_{n-1,k} \circ H_k^{-1}\|_\rho < 1$ this is fulfilled if

$$\|Dh_k\|_{\rho_{k+1}} \cdot \|\psi_{n,k} \circ H_k^{-1} - \psi_{n-1,k} \circ H_k^{-1}\|_\rho < \frac{1}{2^n \cdot \|Dh_1\|_{\rho_1+1} \cdot \dots \cdot \|Dh_{k-1}\|_{\rho_{k-1}+1}}.$$

By our assumption on $\|\psi_{n,k} \circ H_k^{-1} - \psi_{n-1,k} \circ H_k^{-1}\|_\rho$ the claim is true.

In the opposite direction we show that our assumption on $\|h_n \circ R_{\alpha_{n+1}}^m \circ h_n^{-1} - R_{\alpha_n}^m\|_{\rho_{n-1}}$ implies the conditions $\|\psi_{n,k} \circ H_k^{-1} - \psi_{n-1,k} \circ H_k^{-1}\|_\rho < \frac{1}{2^n \cdot \|Dh_1\|_{\rho_1+1} \cdot \dots \cdot \|Dh_k\|_{\rho_{k+1}}}$:

- *Start: $k = n - 1$*

The condition on $\|\psi_{n,n-1} \circ H_{n-1}^{-1} - \psi_{n-1,n-1} \circ H_{n-1}^{-1}\|_\rho \leq \|h_n \circ R_{\alpha_{n+1}}^m \circ h_n^{-1} - R_{\alpha_n}^m\|_{\rho_{n-1}}$ is exactly the supposition of the Lemma.

- *Induction hypothesis:* The claim holds for $2 \leq k \leq n - 1$.

- *Induction step: $k \rightarrow k - 1$*

We estimate with the aid of our induction hypothesis

$$\begin{aligned} \|\psi_{n,k-1} \circ H_{k-1}^{-1} - \psi_{n-1,k-1} \circ H_{k-1}^{-1}\|_\rho &= \|h_k \circ \psi_{n,k} \circ H_k^{-1} - h_k \circ \psi_{n-1,k} \circ H_k^{-1}\|_\rho \\ &\leq \|Dh_k\|_{\rho_{k+1}} \cdot \|\psi_{n,k} \circ H_k^{-1} - \psi_{n-1,k} \circ H_k^{-1}\|_\rho \\ &< \|Dh_k\|_{\rho_{k+1}} \cdot \frac{1}{2^n \cdot \|Dh_1\|_{\rho_1+1} \cdot \dots \cdot \|Dh_k\|_{\rho_{k+1}}} \\ &= \frac{1}{2^n \cdot \|Dh_1\|_{\rho_1+1} \cdot \dots \cdot \|Dh_{k-1}\|_{\rho_{k-1}+1}}. \end{aligned}$$

Hence, the requirements of the fact are met and the Lemma is proven. \square

Now we are able to deduce the aimed statement on convergence of $(f_n)_{n \in \mathbb{N}}$ in the $\text{Diff}_\rho^\omega(\mathbb{T}^2)$ -topology:

Lemma 5.3. *We assume that*

(B1')

$$|\alpha_{n+1} - \alpha_n| < \frac{1}{2^n \cdot \|Dh_1\|_{\rho_1+1} \cdot \dots \cdot \|Dh_{n-1}\|_{\rho_{n-1}+1} \cdot 4\pi n \cdot q_n^{4+\sigma} \cdot \exp(4\pi n \cdot q_n^{1+\sigma} \cdot \tilde{\rho}_{n-1})}.$$

Then the diffeomorphisms $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$ satisfy:

- *The sequence $(f_n)_{n \in \mathbb{N}}$ converges in the $\text{Diff}_\rho^\omega(\mathbb{T}^2)$ -topology to a measure-preserving diffeomorphism f .*
- *We have for every $n \in \mathbb{N}$ and $m \leq q_{n+1}$:*

$$d_0(f^m, f_n^m) < \frac{1}{2^n}.$$

Proof. At first we introduce for $m \in \mathbb{N}$ the function

$$T_m(z) = \cos(2\pi \cdot q_n \cdot (z + m \cdot \alpha_{n+1})) - \cos(2\pi \cdot q_n \cdot z)$$

and exploiting the relation $\cos(x) - \cos(y) = 2 \cdot \sin\left(\frac{x+y}{2}\right) \cdot \sin\left(\frac{y-x}{2}\right)$ we can estimate for every $s \geq 0$:

$$\begin{aligned} \|T_m\|_s &= \|2 \cdot \sin(\pi \cdot q_n \cdot (2z + m \cdot \alpha_{n+1})) \cdot \sin(\pi \cdot q_n \cdot m \cdot \alpha_{n+1})\|_s \\ &= 2 \cdot \left\| \frac{1}{2i} \left(e^{i\pi \cdot q_n \cdot (2z + m \cdot \alpha_{n+1})} - e^{-i\pi \cdot q_n \cdot (2z + m \cdot \alpha_{n+1})} \right) \right\|_s \cdot |\sin(\pi \cdot q_n \cdot m \cdot \alpha_{n+1})| \\ &\leq 2 \cdot \|e^{2\pi i \cdot q_n \cdot z}\|_s \cdot |\sin(\pi \cdot q_n \cdot m \cdot (\alpha_{n+1} - \alpha_n))| \\ &\leq 2 \cdot \|e^{2\pi i \cdot q_n \cdot z}\|_s \cdot \pi \cdot q_n \cdot m \cdot |\alpha_{n+1} - \alpha_n|, \end{aligned}$$

where we made use of $|\sin(x)| \leq |x|$ in the last step.

Using this map T_m we compute

$$\begin{aligned} h_n \circ R_{\alpha_{n+1}}^m \circ h_n^{-1}(\theta, r) &= h_n \circ R_{\alpha_{n+1}}^m(\theta - [nq_n^\sigma] \cdot r, r - q_n^2 \cdot \cos(2\pi \cdot q_n \cdot (\theta - [nq_n^\sigma] \cdot r))) \\ &= h_n(\theta + m \cdot \alpha_{n+1} - [nq_n^\sigma] \cdot r, r - q_n^2 \cdot \cos(2\pi \cdot q_n \cdot (\theta - [nq_n^\sigma] \cdot r))) \\ &= g_n(\theta + m \cdot \alpha_{n+1} - [nq_n^\sigma] \cdot r, r + q_n^2 \cdot T_m(\theta - [nq_n^\sigma] \cdot r)) \\ &= (\theta + m \cdot \alpha_{n+1} + [nq_n^\sigma] \cdot q_n^2 \cdot T_m(\theta - [nq_n^\sigma] \cdot r), r + q_n^2 \cdot T_m(\theta - [nq_n^\sigma] \cdot r)). \end{aligned}$$

Then we have

$$\begin{aligned} &h_n \circ R_{\alpha_{n+1}}^m \circ h_n^{-1}(\theta, r) - R_{\alpha_n}^m(\theta, r) \\ &= (m \cdot \alpha_{n+1} - m \cdot \alpha_n + [nq_n^\sigma] \cdot q_n^2 \cdot T_m(\theta - [nq_n^\sigma] \cdot r), q_n^2 \cdot T_m(\theta - [nq_n^\sigma] \cdot r)). \end{aligned}$$

Thus, we can estimate for $m \leq q_n$:

$$\begin{aligned} &\left\| h_n \circ R_{\alpha_{n+1}}^m \circ h_n^{-1} \circ H_{n-1}^{-1} - R_{\alpha_n}^m \circ H_{n-1}^{-1} \right\|_\rho \leq \left\| h_n \circ R_{\alpha_{n+1}}^m \circ h_n^{-1} - R_{\alpha_n}^m \right\|_{\rho_{n-1}} \\ &\leq 2 \cdot [nq_n^\sigma] \cdot q_n^2 \cdot \|T_m(\theta - [nq_n^\sigma] \cdot r)\|_{\rho_{n-1}} \\ &\leq 2 \cdot [nq_n^\sigma] \cdot q_n^2 \cdot 2 \cdot \left\| e^{2\pi i \cdot q_n \cdot (\theta - [nq_n^\sigma] \cdot r)} \right\|_{\rho_{n-1}} \cdot \pi \cdot q_n \cdot m \cdot |\alpha_{n+1} - \alpha_n| \\ &\leq 4 \cdot \pi \cdot n \cdot q_n^{3+\sigma} \cdot e^{4\pi \cdot n \cdot q_n^{1+\sigma} \cdot \rho_{n-1}} \cdot q_n \cdot |\alpha_{n+1} - \alpha_n| \\ &< \frac{1}{2^n \cdot \|Dh_1\|_{\rho_1+1} \cdot \dots \cdot \|Dh_{n-1}\|_{\rho_{n-1}+1}}. \end{aligned}$$

So the prerequisites of Lemma 5.2 are fulfilled and we conclude $d_\rho(f_n^m, f_{n-1}^m) < \frac{1}{2^n}$. In the same spirit as in the proof of Lemma 4.5 we can show the convergence of $(f_n)_{n \in \mathbb{N}}$ and the second property. \square

Now we formulate the next requirement on the sequence $(q_n)_{n \in \mathbb{N}}$:

$$(B4') \quad q_{n+1} \geq \|Dh_1\|_{\rho_1+1} \cdot \dots \cdot \|Dh_{n-1}\|_{\rho_{n-1}+1} \cdot \|Dh_n\|_{\rho_n+1}.$$

By the requirement $\|Dh_1\|_{\rho_1+1} \cdot \dots \cdot \|Dh_{n-1}\|_{\rho_{n-1}+1} \leq q_n$ as well as equation 6 condition B4' is satisfied if we demand

$$(B4) \quad q_{n+1} \geq 4\pi \cdot n \cdot q_n^{4+\sigma} \cdot \exp(4\pi \cdot q_n^3 \cdot \exp(2\pi \cdot q_n^2 \cdot (1 + n \cdot q_n^\sigma))).$$

Since $|\alpha_{n+1} - \alpha_n| = \frac{a_n}{q_n \cdot \tilde{q}_{n+1}} \leq \frac{1}{\tilde{q}_{n+1}}$ condition B1' yields the requirement

$$\tilde{q}_{n+1} \geq 2^n \cdot \|Dh_1\|_{\rho_1+1} \cdot \dots \cdot \|Dh_{n-1}\|_{\rho_{n-1}+1} \cdot 4\pi \cdot n \cdot q_n^{4+\sigma} \cdot \exp(4\pi \cdot n \cdot q_n^{1+\sigma} \cdot \tilde{\rho}_{n-1}).$$

Using $\tilde{\rho}_{n-1} \leq q_n$ (see condition B2') and $\|Dh_1\|_{\rho_1+1} \cdot \dots \cdot \|Dh_{n-1}\|_{\rho_{n-1}+1} \leq q_n$ (see condition B4') this requirement is satisfied if we demand

$$(B1) \quad \tilde{q}_{n+1} \geq 2^n \cdot 4\pi \cdot n \cdot q_n^{5+\sigma} \cdot \exp(4\pi \cdot n \cdot q_n^{2+\sigma}).$$

By collecting all the prerequisites B1, B2, B3, B4 on the sequence $(q_n)_{n \in \mathbb{N}}$ we demand:

$$q_{n+1} \geq 2^n \cdot 64\pi^2 \cdot n^2 \cdot q_n^8 \cdot \exp(4\pi \cdot n \cdot q_n^3 \cdot \exp(2\pi \cdot q_n^2 \cdot (1 + n \cdot q_n^\sigma))).$$

Since $\tilde{q}_{n+1} = \frac{q_{n+1}}{q_n}$, $q_n = q_{n-1} \cdot \tilde{q}_n \leq \tilde{q}_n^2$ and $0 < \sigma < \frac{1}{2}$ we obtain the following sufficient condition on the growth rate of the rigidity sequence $(\tilde{q}_n)_{n \in \mathbb{N}}$:

$$\tilde{q}_{n+1} \geq 2^n \cdot 64\pi^2 \cdot n^2 \cdot \tilde{q}_n^{14} \cdot \exp(4\pi \cdot n \cdot \tilde{q}_n^6 \cdot \exp(2\pi \cdot \tilde{q}_n^4 \cdot (1 + n \cdot \tilde{q}_n))).$$

6 Proof of weak mixing

By the same approach as in [FS05] we want to apply Proposition 3.4. For this purpose, we introduce a sequence $(m_n)_{n \in \mathbb{N}}$ of natural numbers $m_n \leq q_{n+1}$ in subsection 6.1 and a sequence $(\eta_n)_{n \in \mathbb{N}}$ of standard partial decompositions in subsection 6.2. Finally, we show that the diffeomorphism $\Phi_n := \phi_n \circ R_{\alpha_{n+1}}^{m_n} \circ \phi_n^{-1} \left(\frac{1}{nq_n}, 0, \frac{1}{n} \right)$ -distributes the elements of this partition.

6.1 Choice of the mixing sequence $(m_n)_{n \in \mathbb{N}}$

By condition A3 resp. B3 our chosen sequence $(q_n)_{n \in \mathbb{N}}$ satisfies

$$(C2) \quad q_{n+1} \geq q_n^8.$$

Define

$$\begin{aligned} m_n &= \min \left\{ m \leq q_{n+1} \quad : \quad m \in \mathbb{N}, \quad \inf_{k \in \mathbb{Z}} \left| m \cdot \frac{p_{n+1}}{q_{n+1}} - \frac{1}{2} + \frac{k}{q_n} \right| \leq \frac{q_n}{q_{n+1}} \right\} \\ &= \min \left\{ m \leq q_{n+1} \quad : \quad m \in \mathbb{N}, \quad \inf_{k \in \mathbb{Z}} \left| m \cdot \frac{q_n \cdot p_{n+1}}{q_{n+1}} - \frac{1}{2} + k \right| \leq \frac{q_n^2}{q_{n+1}} \right\} \end{aligned}$$

Lemma 6.1. *The set $M_n := \left\{ m \leq q_{n+1} \quad : \quad m \in \mathbb{N}, \quad \inf_{k \in \mathbb{Z}} \left| m \cdot \frac{q_n \cdot p_{n+1}}{q_{n+1}} - \frac{1}{2} + k \right| \leq \frac{q_n^2}{q_{n+1}} \right\}$ is nonempty for every $n \in \mathbb{N}$, i.e. m_n exists.*

Proof. The number α_{n+1} was constructed by the rule $\frac{p_{n+1}}{q_{n+1}} = \frac{p_n}{q_n} - \frac{a_n}{q_n \cdot \tilde{q}_{n+1}}$, where $a_n \in \mathbb{Z}$, $1 \leq a_n \leq q_n$, i.e. $p_{n+1} = p_n \cdot \tilde{q}_{n+1} - a_n$ and $q_{n+1} = q_n \cdot \tilde{q}_{n+1}$. So $\frac{q_n \cdot p_{n+1}}{q_{n+1}} = \frac{p_n}{\tilde{q}_{n+1}}$ and the set $\left\{ j \cdot \frac{q_n \cdot p_{n+1}}{q_{n+1}} \quad : \quad j = 1, 2, \dots, q_{n+1} \right\}$ contains $\frac{\tilde{q}_{n+1}}{\gcd(p_{n+1}, \tilde{q}_{n+1})}$ different equally distributed points on \mathbb{S}^1 . Hence, there are at least $\frac{\tilde{q}_{n+1}}{q_n} = \frac{q_{n+1}}{q_n^2}$ different such points and so for every $x \in \mathbb{S}^1$ there is a $j \in \{1, \dots, q_{n+1}\}$ such that

$$\inf_{k \in \mathbb{Z}} \left| x - j \cdot \frac{q_n \cdot p_{n+1}}{q_{n+1}} + k \right| \leq \frac{q_n^2}{q_{n+1}}.$$

In particular, this is true for $x = \frac{1}{2}$. □

Remark 6.2. We define

$$\Delta_n = \left(m_n \cdot \frac{p_{n+1}}{q_{n+1}} - \frac{1}{2 \cdot q_n} \right) \bmod \frac{1}{q_n}.$$

By the above construction of m_n it holds: $|\Delta_n| \leq \frac{q_n}{q_{n+1}}$. By C2 we get:

$$|\Delta_n| \leq \frac{1}{q_n^7}.$$

6.2 Standard partial decomposition η_n

In the following we will consider the set

$$B_n = \bigcup_{k=0}^{2q_n-1} \left[\frac{k}{2q_n} - \frac{1}{2q_n^{1.5}}, \frac{k}{2q_n} + \frac{1}{2q_n^{1.5}} \right].$$

Our horizontal intervals belonging to the partial decomposition η_n will lie outside B_n . To show that $\Phi_n \left(\frac{1}{nq_n}, 0, \frac{1}{n} \right)$ -distributes the elements of this partition we will need the subsequent result similar to the concept of “uniformly stretching” from [Fa02]:

Lemma 6.3. *Let $I = [a, b] \subset \mathbb{R}$ be an interval and $\psi : I \rightarrow \mathbb{R}$ be a strictly monotonic C^2 -function. Furthermore, we denote $J := [\inf_{x \in I} \psi(x), \sup_{x \in I} \psi(x)]$. If ψ satisfies*

$$\sup_{x \in I} |\psi''(x)| \cdot \lambda(I) \leq \varepsilon \cdot \inf_{x \in I} |\psi'(x)|,$$

then for any interval $\tilde{J} \subseteq J$ we have

$$\left| \frac{\lambda(I \cap \psi^{-1}(\tilde{J}))}{\lambda(I)} - \frac{\lambda(\tilde{J})}{\lambda(J)} \right| \leq \varepsilon \cdot \frac{\lambda(\tilde{J})}{\lambda(J)}.$$

Proof. We consider the case that ψ is strictly increasing (the proof in the decreasing case is analogous), which implies $\psi' > 0$ (due to our assumption $\sup_{x \in I} |\psi''(x)| \cdot \lambda(I) \leq \varepsilon \cdot \inf_{x \in I} |\psi'(x)|$) and $J = [\psi(a), \psi(b)]$.

Let $\tilde{J} = [\psi(c), \psi(d)]$, where $a \leq c \leq d \leq b$. By the mean value theorem there are $\xi_1 \in [a, b]$ and $\xi_2 \in [c, d]$, such that $\psi(b) - \psi(a) = \psi'(\xi_1) \cdot (b - a)$ resp. $\psi(d) - \psi(c) = \psi'(\xi_2) \cdot (d - c)$. Applying the mean value theorem on ψ' gives $\xi_3 \in [a, b]$ with $|\psi'(\xi_2) - \psi'(\xi_1)| = |\psi''(\xi_3)| \cdot |\xi_2 - \xi_1|$. Then we have:

$$|\psi'(\xi_1) - \psi'(\xi_2)| \leq \sup_{x \in [a, b]} |\psi''(x)| \cdot |b - a| = \sup_{x \in [a, b]} |\psi''(x)| \cdot \lambda(I) \leq \varepsilon \cdot \inf_{x \in [a, b]} |\psi'(x)| \leq \varepsilon \cdot |\psi'(\xi_2)|.$$

Hereby, we obtain:

$$\left| \frac{\psi'(\xi_1)}{\psi'(\xi_2)} - 1 \right| \leq \varepsilon.$$

Since $\psi' > 0$ this implies $1 - \varepsilon \leq \frac{\psi'(\xi_1)}{\psi'(\xi_2)} \leq 1 + \varepsilon$ and thus:

$$\frac{\lambda(I \cap \psi^{-1}(\tilde{J}))}{\lambda(I)} = \frac{d - c}{b - a} = \frac{\psi'(\xi_1) \cdot (\psi(d) - \psi(c))}{\psi'(\xi_2) \cdot (\psi(b) - \psi(a))} \leq (1 + \varepsilon) \cdot \frac{\lambda(\tilde{J})}{\lambda(J)}.$$

This implies

$$\frac{\lambda\left(I \cap \psi^{-1}\left(\tilde{J}\right)\right)}{\lambda(I)} - \frac{\lambda\left(\tilde{J}\right)}{\lambda(J)} \leq \varepsilon \cdot \frac{\lambda\left(\tilde{J}\right)}{\lambda(J)}.$$

Similarly we obtain the estimate from below and the claim follows. \square

By the explicit definitions of the conjugation maps the transformation $\Phi_n = \phi_n \circ R_{\alpha_{n+1}}^{m_n} \circ \phi_n^{-1}$ has a lift of the form

$$\Phi_n(\theta, r) = (\theta + m_n \cdot \alpha_{n+1}, r + \psi_n(\theta))$$

with

$$\psi_n(\theta) = q_n^2 \cdot (\cos(2\pi(q_n\theta + m_n q_n \alpha_{n+1})) - \cos(2\pi q_n \theta)).$$

We examine this map ψ_n :

Lemma 6.4. *The map ψ_n satisfies:*

$$\inf_{\theta \in \mathbb{T} \setminus B_n} |\psi'_n(\theta)| \geq q_n^{2.5} \quad \text{and} \quad \sup_{\theta \in \mathbb{T} \setminus B_n} |\psi''_n(\theta)| \leq 9\pi^2 q_n^4.$$

Proof. Using the relation $\cos(2\pi q_n \theta + \pi) = -\cos(2\pi q_n \theta)$ and the map

$$\sigma_n(\theta) = q_n^2 \cdot \left(\cos(2\pi q_n(\theta + m_n \alpha_{n+1})) - \cos\left(2\pi q_n\left(\theta + \frac{1}{2q_n}\right)\right) \right)$$

we can write $\psi_n(\theta) = -2q_n^2 \cdot \cos(2\pi q_n \theta) + \sigma_n(\theta)$. For this map σ_n we compute:

$$\sigma'_n(\theta) = 2\pi \cdot q_n^3 \cdot \left(-\sin(2\pi q_n(\theta + m_n \cdot \alpha_{n+1})) + \sin\left(2\pi q_n\left(\theta + \frac{1}{2q_n}\right)\right) \right).$$

Applying the mean value theorem on the function $\varphi_a(\xi) := -2\pi \cdot q_n^3 \cdot \sin(2\pi q_n \xi)$ we obtain:

$$|\sigma'_n(\theta)| \leq 2\pi q_n^3 \cdot \max_{\xi \in \mathbb{T}} | -2\pi q_n \cdot \cos(2\pi q_n \xi) | \cdot \Delta_n \leq (2\pi)^2 \cdot q_n^4 \cdot \frac{q_n}{q_{n+1}} \leq (2\pi)^2 \cdot q_n^4 \cdot \frac{1}{q_n^7} < 1.$$

On the other hand, on the set $\mathbb{T} \setminus B_n$ it holds:

$$\begin{aligned} \inf_{\theta \in \mathbb{T} \setminus B_n} |\sin(2\pi q_n \theta)| &= \inf_{\theta = \tilde{\theta} + \frac{k}{2q_n}, k \in \mathbb{Z}, \tilde{\theta} \in \left[\frac{1}{2q_n^{1.5}}, \frac{1}{2q_n} - \frac{1}{2q_n^{1.5}} \right]} |\sin(2\pi q_n \theta)| \\ &= \inf_{\tilde{\theta} \in \left[\frac{1}{2q_n^{1.5}}, \frac{1}{2q_n} - \frac{1}{2q_n^{1.5}} \right]} \left| \sin\left(2\pi q_n \tilde{\theta}\right) \right| \\ &= \inf_{\tilde{\theta} \in \left[\frac{1}{2q_n^{1.5}}, \frac{1}{4q_n} \right]} \left| \sin\left(2\pi q_n \tilde{\theta}\right) \right| \geq \frac{1}{2} \cdot 2\pi q_n \cdot \frac{1}{2q_n^{1.5}} = \frac{\pi}{2} \cdot q_n^{-0.5} > q_n^{-0.5} \end{aligned}$$

with the aid of the estimate $\sin(x) \geq \frac{1}{2}x$ for $x \in [0, \frac{\pi}{2}]$. Thus, we have:

$$\inf_{\theta \in \mathbb{T} \setminus B_n} |\psi'_n(\theta)| \geq 4\pi q_n^3 \cdot \inf_{\theta \in \mathbb{T} \setminus B_n} |\sin(2\pi q_n \theta)| - \sup_{\theta \in \mathbb{T} \setminus B_n} |\sigma'_n(\theta)| \geq 4\pi q_n^3 \cdot q_n^{-0.5} - 1 \geq q_n^{2.5}.$$

In order to estimate ψ''_n we compute

$$\sigma''_n(\theta) = (2\pi)^2 \cdot q_n^4 \cdot \left(-\cos(2\pi q_n(\theta + m_n \cdot \alpha_{n+1})) + \cos\left(2\pi q_n\left(\theta + \frac{1}{2q_n}\right)\right) \right)$$

and use the mean value theorem on $\varphi_b(\xi) := -(2\pi)^2 \cdot q_n^4 \cdot \cos(2\pi q_n \xi)$:

$$|\sigma_n''(\theta)| \leq (2\pi)^2 \cdot q_n^4 \cdot \max_{\xi \in \mathbb{T}} |2\pi q_n \cdot \sin(2\pi q_n \xi)| \cdot \Delta_n \leq (2\pi)^3 \cdot q_n^5 \cdot \frac{q_n}{q_{n+1}} < 1.$$

Then we obtain:

$$\sup_{\theta \in \mathbb{T} \setminus B_n} |\psi_n''(\theta)| \leq \sup_{\theta \in \mathbb{T} \setminus B_n} \left| 2 \cdot (2\pi)^2 \cdot q_n^4 \cdot \cos(2\pi q_n \theta) \right| + \sup_{\theta \in \mathbb{T} \setminus B_n} |\sigma_n''(\theta)| \leq 8\pi^2 \cdot q_n^4 + 1 \leq 9\pi^2 \cdot q_n^4.$$

□

The aimed standard partial decomposition η_n of \mathbb{T}^2 is defined in the following way:
Let $\hat{\eta}_n = \{\hat{I}_{n,i}\}$ be the partial partition of $\mathbb{T} \setminus B_n$ consisting of all the disjoint intervals $\hat{I}_{n,i}$ such that there is $k \in \mathbb{Z}$: $\psi_n(\hat{I}_{n,i}) = k + [0, 1)$. Then we define

$$\eta_n = \left\{ \hat{I} \times \{r\} : \hat{I} \in \hat{\eta}_n, r \in \mathbb{T} \right\}.$$

Note that we have $\pi_r(\Phi_n(I_n)) = \mathbb{T}$ for every $I_n \in \eta_n$.

Lemma 6.5. *For any partition element $\hat{I}_n \in \hat{\eta}_n$ we have $\lambda(\hat{I}_n) \leq q_n^{-2.5}$. Moreover, it holds: $\eta_n \rightarrow \varepsilon$.*

Proof. By Lemma 6.4 we have $\inf_{\theta \in \mathbb{T} \setminus B_n} |\psi_n'(\theta)| \geq q_n^{2.5}$. Therefore, $\lambda(\hat{I}_n) \leq q_n^{-2.5}$ for any $I_n \in \eta_n$. Hence, the length of the elements of η_n goes to zero as $n \rightarrow \infty$. Thus, in order to prove $\eta_n \rightarrow \varepsilon$ we have only to check that the total measure of the partial decompositions η_n goes to 1 as $n \rightarrow \infty$. Since the elements of $\hat{\eta}_n$ are contained in $\mathbb{T} \setminus B_n$ and have to satisfy the additional requirement $\psi_n(\hat{I}_n) = k + [0, 1)$ for $k \in \mathbb{Z}$, on both sides around the set B_n there is an area without partition elements. So the total measure of η_n can be estimated as follows:

$$\begin{aligned} \sum_{\hat{I}_n \in \hat{\eta}_n} \lambda(\hat{I}_n) &\geq 1 - \lambda(B_n) - 2 \cdot 2q_n \cdot \max_{\hat{I}_n \in \hat{\eta}_n} \lambda(\hat{I}_n) \\ &\geq 1 - 2q_n \cdot (q_n^{-1.5} + 2q_n^{-2.5}) > 1 - 3 \cdot q_n^{-0.5} \end{aligned}$$

and this approaches 1 as $n \rightarrow \infty$.

□

6.3 Application of the criterion for weak mixing

In order to apply the criterion for weak mixing we check that the constructed diffeomorphism $f = \lim_{n \rightarrow \infty} f_n$ fulfil the requirements:

- By Lemma 4.5, 2., resp. 5.3, 2.: $d_0(f^{m_n}, f_n^{m_n}) < \frac{1}{2^n}$, because $m_n \leq q_{n+1}$ by definition.
- Because of the requirement A3 resp. B3 on the number q_n we have C1.
- By Lemma 6.5 we have $\eta_n \rightarrow \varepsilon$ and the length of the horizontal interval is at most $q_n^{-2.5}$.

Finally, the next Lemma proves the last remaining property:

Lemma 6.6. *Let $I_n \in \eta_n$. Then $\Phi_n\left(\frac{1}{nq_n}, 0, \frac{1}{n}\right)$ -distributes I_n .*

Proof. The partial partition η_n was chosen in such a way that $\pi_r(\Phi_n(I_n)) = \mathbb{T}$. Hence, δ can be taken equal to 0.

Using the form of the lift $\Phi_n(\theta, r) = (\theta + m_n \cdot \alpha_{n+1}, r + \psi_n(\theta))$ we observe that $\Phi_n(I_n)$ is contained in the vertical strip $(I_n + m_n \cdot \alpha_{n+1}) \times \mathbb{T}$ for every $I_n \in \eta_n$. Since the length of I_n is at most $\frac{1}{q_n^{2.5}} < \frac{1}{nq_n}$ by Lemma 6.5, we can take $\gamma = \frac{1}{nq_n}$.

Recall that an element $I_n \in \eta_n$ has the form $\hat{I} \times \{r\}$ for some $r \in \mathbb{T}$ and an interval $\hat{I} \in \hat{\eta}_n$ contained in $\mathbb{T} \setminus B_n$ with $\lambda(\hat{I}) \leq q_n^{-2.5}$ (see Lemma 6.5). Then Lemma 6.4 implies the estimate

$$\frac{\sup_{\theta \in \hat{I}} |\psi_n''(\theta)|}{\inf_{\theta \in \hat{I}} |\psi_n'(\theta)|} \cdot \lambda(\hat{I}) \leq \frac{9\pi^2 q_n^4}{q_n^{2.5}} \cdot q_n^{-2.5} = \frac{9\pi^2}{q_n} < \frac{1}{n}.$$

Then we can apply Lemma 6.3 on ψ_n and \hat{I} with $\varepsilon = \frac{1}{n}$. Moreover, we note that for any $\tilde{J} \subseteq J = \mathbb{T}$ the fact $\Phi_n(\theta, r) \in \mathbb{T} \times \tilde{J}$ is equivalent to $\psi_n(\theta) \in \tilde{J} - r := \{j - r : j \in \tilde{J}\}$. Combining these both results we conclude:

$$\left| \frac{\lambda(I_n \cap \Phi_n^{-1}(\mathbb{T} \times \tilde{J}))}{\lambda(I_n)} - \frac{\lambda(\tilde{J})}{\lambda(J)} \right| = \left| \frac{\lambda(\hat{I} \cap \psi_n^{-1}(\tilde{J} - r))}{\lambda(\hat{I})} - \frac{\lambda(\tilde{J})}{\lambda(J)} \right| \leq \frac{1}{n} \cdot \frac{\lambda(\tilde{J})}{\lambda(J)}.$$

Thus, we can choose $\varepsilon = \frac{1}{n}$ in the definition of $(\gamma, \delta, \varepsilon)$ -distribution. \square

Then we can apply Proposition 3.4 and conclude that the constructed diffeomorphisms are weak mixing.

7 Proof of the Corollaries

By some estimates we show that the assumptions of the Corollaries are enough to fulfil the requirements of the corresponding theorem.

7.1 Proof of Corollary 1

We recall the assumptions $\tilde{q}_1 \geq 108\pi$ and $\tilde{q}_{n+1} \geq \tilde{q}_n^{\tilde{q}_n}$ on the sequence $(\tilde{q}_n)_{n \in \mathbb{N}}$.

Claim: Under these assumptions the numbers \tilde{q}_n satisfy $\tilde{q}_n \geq 4\pi \cdot n \cdot (n+2)^{n+2}$.

Proof with the aid of complete induction:

- *Start $n = 1$:* $\tilde{q}_1 \geq 108\pi = 4\pi \cdot (1+2)^{1+2}$
- *Assumption:* The claim is true for $n \in \mathbb{N}$.
- *Induction step $n \rightarrow n+1$:* We calculate

$$\begin{aligned} \tilde{q}_{n+1} &\geq \tilde{q}_n^{\tilde{q}_n} \geq \left(4\pi \cdot n \cdot (n+2)^{n+2}\right)^{4\pi \cdot n \cdot (n+2)^{n+2}} \\ &\geq 4\pi \cdot (n \cdot (n+2))^{\pi \cdot n \cdot (n+2)^{n+2}} \cdot \left((n+2)^{n+1}\right)^{3\pi \cdot n \cdot (n+2)^{n+2}} \\ &\geq 4\pi \cdot (n+1) \cdot (n+3)^{n+3}, \end{aligned}$$

where we used the relation $(n+2)^{n+1} \geq n+3$ in the last step.

Hereby, we have

$$\begin{aligned}
\tilde{q}_{n+1} &\geq \tilde{q}_n^{\tilde{q}_n} \geq \tilde{q}_n^{4\pi \cdot n \cdot (n+2)^{n+2}} = \tilde{q}_n^{\pi \cdot n \cdot (n+2)^{n+2}} \cdot \tilde{q}_n^{3\pi \cdot n \cdot (n+2)^{n+2}} \\
&\geq \left(4\pi \cdot n \cdot (n+2)^{n+2}\right)^{\pi \cdot n \cdot (n+2)^{n+2}} \cdot \tilde{q}_n^{2 \cdot (n+2)^{n+2}} \\
&\geq (4\pi \cdot n \cdot (n+2)!)^{(n+2)^{n+2}} \cdot \tilde{q}_n^{2 \cdot ((n+2) \cdot (n+1)^{n+1} + 1)} \\
&\geq 2^n \cdot (2\pi \cdot n \cdot (n+2)!)^{(n+2)^{n+1} \cdot (n+1)} \cdot \tilde{q}_n^{2 \cdot ((n+2) \cdot (n+1)^{n+1} + 1)} \\
&\geq 2^n \cdot (2\pi n)^{(n+2) \cdot (n+1)^{n+1}} \cdot (n+2)! \cdot ((n+2)!)^{(n+2)^{n-1} \cdot (n+1)} \cdot \tilde{q}_n^{2 \cdot ((n+2) \cdot (n+1)^{n+1} + 1)} \\
&\geq \varphi_1(n) \cdot \tilde{q}_n^{2 \cdot ((n+2) \cdot (n+1)^{n+1} + 1)}
\end{aligned}$$

Hence, the requirement of Theorem 1 is met.

7.2 Proof of Corollary 2

Let $(\tilde{q}_n)_{n \in \mathbb{N}}$ be a sequence satisfying $\tilde{q}_1 \geq (\rho + 1) \cdot 2^7 \cdot \pi^2$ and $\tilde{q}_{n+1} \geq \tilde{q}_n^{15} \cdot \exp(\tilde{q}_n^7 \cdot \exp(\tilde{q}_n^6))$. Again we start with a proof by complete induction:

Claim: The numbers \tilde{q}_n satisfy $\tilde{q}_n \geq 2^{n+6} \cdot n^2 \cdot \pi^2$.

Proof:

- *Start $n = 1$:* By assumption we have: $\tilde{q}_1 \geq (\rho + 1) \cdot 2^7 \cdot \pi^2 \geq 2^{1+6} \cdot 1^2 \cdot \pi^2$.
- *Assumption:* The claim is true for $n \in \mathbb{N}$.
- *Induction step $n \rightarrow n + 1$:* We estimate

$$\tilde{q}_{n+1} \geq \tilde{q}_n^{15} \cdot \exp(\tilde{q}_n^7 \cdot \exp(\tilde{q}_n^6)) \geq (2^{n+6} \cdot n^2 \cdot \pi^2)^{15} \cdot \exp(\tilde{q}_n^7 \cdot \exp(\tilde{q}_n^6)) \geq 2^{n+7} \cdot (n+1)^2 \cdot \pi^2.$$

Then we have:

$$\begin{aligned}
\tilde{q}_{n+1} &\geq \tilde{q}_n^{15} \cdot \exp(\tilde{q}_n^7 \cdot \exp(\tilde{q}_n^6)) \\
&\geq 2^{n+6} \cdot n^2 \cdot \pi^2 \cdot \tilde{q}_n^{14} \cdot \exp(2^{n+6} \cdot n^2 \cdot \pi^2 \cdot \tilde{q}_n^6 \cdot \exp(2^{n+6} \cdot n^2 \cdot \pi^2 \cdot \tilde{q}_n^5)) \\
&\geq 2^n \cdot n^2 \cdot 64\pi^2 \cdot \tilde{q}_n^{14} \cdot \exp(4\pi \cdot n \cdot \tilde{q}_n^6 \cdot \exp(2\pi \cdot 2 \cdot n \cdot \tilde{q}_n^5)).
\end{aligned}$$

Thus, the condition of Theorem 2 is fulfilled.

References

- [AK70] D. V. Anosov and A. Katok: *New examples in smooth ergodic theory. Ergodic diffeomorphisms*. Trudy Moskov. Mat. Obsc., 23: 3 - 36, 1970.
- [BJLR] V. Bergelson, A. Del Junco, M. Lemanczyk and J. Rosenblatt: *Rigidity and non-recurrence along sequences*. To appear in Ergodic Theory and Dynamical Systems.
- [Fa02] B. Fayad: *Analytic mixing reparametrizations of irrational flows*. Ergodic Theory Dynamical Systems, 22: 437 - 468, 2002.

- [FK04] B. Fayad and A. Katok: *Constructions in elliptic dynamics*. Ergodic Theory Dynam. Systems, 24 (5): 1477 - 1520, 2004.
- [FS05] B. Fayad and M. Saprykina: *Weak mixing disc and annulus diffeomorphisms with arbitrary Liouville rotation number on the boundary*. Ann. Scient. École. Norm. Sup.(4), 38(3): 339 - 364, 2005.
- [FSW07] B. Fayad, M. Saprykina and A. Windsor: *Nonstandard smooth realizations of Liouville rotations*. Ergodic Theory Dynam. Systems, 27: 1803 - 1818, 2007.
- [GK00] R. Gunesch and A. Katok: *Construction of weakly mixing diffeomorphisms preserving measurable Riemannian metric and smooth measure*. Discrete Contin. Dynam. Systems, 6: 61- 88, 2000.
- [GM89] S. Glasner and D. Maon: *Rigidity in topological dynamics*. Ergodic Theory Dynam. Systems, 9: 309 - 320, 1989.
- [JKLSS09] J. James, T. Koberda, K. Lindsey, C. Silva and P. Speh: *On ergodic transformations that are both weak mixing and uniformly rigid*, New York J. Math, 15: 393 - 403, 2009
- [Ly99] M. Lyubich: *Feigenbaum-Couillet-Tresser universality and Milnor's hairiness conjecture*. Ann. of Math., 149, no. 2: 311 - 326, 1988.
- [Sa03] M. Saprykina: *Analytic non-linearizable uniquely ergodic diffeomorphisms on \mathbb{T}^2* . Ergodic Theory Dynamical Systems, 23: 935 - 955, 2003.
- [Ya13] K. Yancey: *On weakly mixing homeomorphisms of the two-torus that are uniformly rigid*. Journal of Mathematical Analysis and Applications, 399 (2): 524-541, 2013.

Philipp Kunde
University of Hamburg
Bundesstraße 55, 20146 Hamburg, Germany
Email: Philipp.Kunde@math.uni-hamburg.de